Karush-Kuhn-Tucker necessary conditions for local Pareto minima of constrained multiobjective programming problems

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Abstract Under the constraint qualification of Abadie type we establish necessary efficiency conditions described by inconsistent inequalities, Karush-Kuhn-Tucker necessary conditions and strong Karush-Kuhn-Tucker necessary conditions for local Pareto minima of nonsmooth multiobjective programming problems involving inequality, equality and set constraints in Banach spaces in terms of convexificators. Note that all the constraint functions involving in the considering problem are not necessarily continuous, except inactive constraints.

1 Introduction

The Karush-Kuhn-Tucker conditions of nonsmooth multiobjective programming problems is a significat topic in optimization. If Lagrange multipliers corresponding to all the components of the objective function are positive, they are called strong Karush-Kuhn-Tucker conditions. The Lagrange multiplier rules in terms of different subdifferentials for nonsmooth optimization problems have been studied by many authors (see, e. g., [4], [6 - 16], and references therein). The notion of convex and compact convexificator was first introduced by Demyanov [2]. This is a generalization of the notions of upper convex and lower concave approximations in [3]. The notions of nonconvex closed convexificator and approximate Jacobian were introduced by Jeyakumar and Luc in [6] and [7], respectively. They have provided good calculus rules for establishing necessary optimality conditions in nonsmooth optimization. The notion of convexificator is a generalization of some notions of known subdifferentials such as the subdifferentials of Clarke [1], Michel-Penot [15], Mordukhovich [16].

In this paper, under the constraint qualifications of Abadie type we establish necessary efficiency conditions described by inconsistent inequalities, Karush-Kuhn-Tucker necessary conditions and strong Karush-Kuhn-Tucker necessary conditions for local Pareto minima of nonsmooth multiobjective programming problems involving inequality, equality and set constraints in Banach spaces in terms of convexificators.
2 Notions and definitions

Let $X$ be a Banach space, $X^*$ topological dual of $X$ and $f$ a extended-real-valued function defined on $X$. The lower (resp. upper) Dini directional derivatives of $f$ at $\mathfrak{x} \in X$ in a direction $v \in X$ are defined, respectively, by

$$f^-(\mathfrak{x}; v) = \liminf_{t \downarrow 0} \frac{f(\mathfrak{x} + tv) - f(\mathfrak{x})}{t},$$

(resp. $f^+ (\mathfrak{x}; v) = \limsup_{t \downarrow 0} \frac{f(\mathfrak{x} + tv) - f(\mathfrak{x})}{t}$).

In case $f^+(\mathfrak{x}; v) = f^-(\mathfrak{x}; v)$, we denote their common value by $f'(\mathfrak{x}; v)$, which is called Dini derivative of $f$ at $\mathfrak{x}$ in the direction $v$. The function $f$ is Dini differentiable at $\mathfrak{x}$ if its Dini derivative at $\mathfrak{x}$ exists in all directions. Recall [6] that the function $f$ is said to have an upper (resp. lower) convexificator $\partial^* f(\mathfrak{x})$ (resp. $\partial^*_0 f(\mathfrak{x})$) at $\mathfrak{x}$ if $\partial^* f(\mathfrak{x})$ (resp. $\partial^*_0 f(\mathfrak{x})$) $\subseteq X^*$ is weak$^*$ closed, and

$$f^-(\mathfrak{x}; v) \leq \sup_{\xi \in \partial^* f(\mathfrak{x})} \langle \xi, v \rangle \quad (\forall v \in X),$$

(resp. $f^+(\mathfrak{x}; v) \geq \inf_{\xi \in \partial^*_0 f(\mathfrak{x})} \langle \xi, v \rangle \quad (\forall v \in X)$).

A weak$^*$ closed set $\partial^* f(\mathfrak{x}) \subseteq X^*$ is said to be a convexificator of $f$ at $\mathfrak{x}$ if it is both upper and lower convexificator of $f$ at $\mathfrak{x}$. Note that upper and lower convexificators are not unique. For a locally Lipschitz function, the Clarke subdifferential and the Michel-Penot subdifferential are convexificators of $f$ at $\mathfrak{x}$ (see [6]). The function $f$ is said to have an upper semi-regular convexificator $\partial^* f(\mathfrak{x})$ at $\mathfrak{x}$ if $\partial^* f(\mathfrak{x}) \subseteq X^*$ is weak$^*$ closed, and

$$f^+(\mathfrak{x}; v) \leq \sup_{\xi \in \partial^* f(\mathfrak{x})} \langle \xi, v \rangle \quad (\forall v \in X).$$

(1)

Following [6], if equality holds in (1) then $\partial^* f(\mathfrak{x})$ is called an upper regular convexificator. For a locally Lipschitz function and regular in the sense of Clarke [1], the Clarke subdifferential is an upper regular convexificator (see [4]).

Example 2.1 Let $f : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) := \begin{cases} x, &\text{if } x \in \mathbb{Q} \cap [0, +\infty[ , \\ x^4 - 4x^3 + 4x^2, &\text{if } x \in \mathbb{Q} \cap ]-\infty, 0], \\ 0, &\text{otherwise}, \end{cases}$$

where $\mathbb{Q}$ is the set of rationals. Then

$$f^+(0; v) = \begin{cases} v, &\text{if } v \geq 0, \\ 0, &\text{if } v < 0, \end{cases}$$
\( f^{-}(0; v) = 0 \ (\forall v \in \mathbb{R}) \).

The set \( \{0; 1\} \) is an upper semi-regular convexificator of \( f \) at \( \overline{x} = 0 \), and so it is upper convexificator of \( f \) at \( \overline{x} \). The set \( \{0\} \) is lower convexificator of \( f \) at \( \overline{x} \).

Recall [1] that the contingent cone to a set \( C \subseteq X \) at a point \( \overline{x} \in C \) is defined as

\[
K(C; \overline{x}) = \left\{ v \in X : \exists v_n \to v, \exists t_n \downarrow 0 \text{ such that } \overline{x} + t_nv_n \in C, \forall n \right\}.
\]

The cone of sequential linear directions (or sequential radial cone) to \( C \) at \( \overline{x} \in C \) is

\[
Z(C; \overline{x}) = \left\{ v \in X : \exists t_n \downarrow 0 \text{ such that } \overline{x} + t_nv \in C, \forall n \right\}.
\]

Note that both these cones are nonempty and \( Z(C; \overline{x}) \subseteq K(C; \overline{x}) \).

### 3 Karush-Kuhn-Tucker necessary conditions for local Pareto minima

Let \( f, g, h \) be mappings from a Banach space \( X \) into \( \mathbb{R}^m, \mathbb{R}^n, \mathbb{R}^\ell \), respectively, and let \( C \) be a subset of \( X \). Then \( f, g, h \) are of the forms: \( f = (f_1, \ldots, f_m), \ g = (g_1, \ldots, g_n), \ h = (h_1, \ldots, h_\ell) \), where \( f_1, \ldots, f_m, g_1, \ldots, g_n, h_1, \ldots, h_\ell \) are extended-real-valued functions defined on \( X \). For the sake of simplicity, we set: \( I = \{1, \ldots, n\}, J = \{1, \ldots, m\} \) and \( L = \{1, \ldots, \ell\} \). We shall be concerned with the following multiobjective programming problem (MP):

\[
\text{min } f(x) \text{ s.t. } x \text{ belongs to } M := \left\{ x \in C : g_i(x) \leq 0, i \in I, h_j(x) = 0, j \in L \right\}.
\]

We set \( I(\overline{x}) = \{i \in I : g_i(\overline{x}) = 0\} \). Recall that a point \( \overline{x} \in M \) is said to be a local Pareto minimizer of Problem (MP) if there exists a number \( \delta > 0 \) such that there is no \( x \in M \cap B(\overline{x}; \delta) \) satisfying

\[
\begin{align*}
f_k(x) &\leq f_k(\overline{x}) \quad (\forall k \in J), \\
 f_s(x) &< f_s(\overline{x}) \quad \text{at least one } s \in J,
\end{align*}
\]

where \( B(\overline{x}; \delta) \) stands for the open ball of radius \( \delta \) around \( \overline{x} \).

For \( \overline{x} \in X \) and \( s \in J \), we set

\[
Q_s(\overline{x}) = \left\{ x \in C : f_k(x) \leq f_k(\overline{x}) \ (\forall k \in J, k \neq s), \ g_j(x) \leq 0 \right. \\
\left. \quad (\forall i \in I(\overline{x}), \ h_j(x) = 0 \ (\forall j \in L) \right\}.
\]

If \( h_j \) is Dini differentiable at \( \overline{x} \) for all \( j \in L \), we put

\[
C(Q_s(\overline{x}); \overline{x}) = \left\{ v \in Z(C; \overline{x}) : f_k^{-}(\overline{x}; v) \leq 0 \ (\forall k \in J, k \neq s), \\
g_i^{-}(\overline{x}; v) \leq 0 \ (\forall i \in I(\overline{x}), \ h_j(\overline{x}; v) = 0 \ (\forall j \in L) \right\}.
\]
The relationship between $Z(Q_s(\bar{x}); \bar{x})$ and $C(Q_s(\bar{x}); \bar{x})$ is shown in the following result.

**Theorem 3.1** Let $\bar{x} \in M$ and $h_j$ is Dini differentiable at $\bar{x}$ for all $j \in L$. Then, for $s \in J$,

$$Z(Q_s(\bar{x}); \bar{x}) \subseteq C(Q_s(\bar{x}); \bar{x}).$$

**Proof** For $v \in Z(Q_s(\bar{x}); \bar{x})$, there exists $t_n \downarrow 0$ such that $\bar{x} + t_n v \in Q_s(\bar{x})$. Hence, $\bar{x} + t_n v \in C$, and

$$f_k(\bar{x} + t_n v) \leq f_k(\bar{x}) \quad (\forall k \in J, k \neq s),$$

$$g_i(\bar{x} + t_n v) \leq 0 = g_i(\bar{x}) \quad (\forall i \in I(\bar{x})),$$

$$h_j(\bar{x} + t_n v) = 0 \quad (\forall j \in L).$$

These reduce to that $v \in Z(C; \bar{x})$, and

$$f_k^-(\bar{x}; v) = \lim_{n \to +\infty} \frac{f_k(\bar{x} + t_n v) - f_k(\bar{x})}{t_n} \leq 0 \quad (\forall k \in J, k \neq s),$$

$$g_i^-(\bar{x}; v) = \lim_{n \to +\infty} \frac{g_i(\bar{x} + t_n v) - g_i(\bar{x})}{t_n} \leq 0 \quad (\forall i \in I(\bar{x})),$$

$$h_j'(\bar{x}; v) = \lim_{n \to +\infty} \frac{h_j(\bar{x} + t_n v) - h_j(\bar{x})}{t_n} = 0 \quad (\forall j \in L).$$

Consequently, $v \in C(Q_s(\bar{x}); \bar{x})$, and so the result follows. \hfill \Box

To derive Kuhn-Tucker necessary conditions for local Pareto minima of Problem (MP), we introduce the following constraint qualification of Abadie type ($CQ_s$):

$$C(Q_s(\bar{x}); \bar{x}) \subseteq Z(Q_s(\bar{x}); \bar{x}).$$

A necessary condition for local Pareto minima of (MP) can be stated as follows.

**Theorem 3.2** Let $\bar{x}$ be a local Pareto minimum of (MP), and let $T$ be an arbitrary nonempty closed convex subcone of $Z(C; \bar{x})$ with vertex at the origin. Assume that the constraint qualification ($CQ_s$) holds for some $s \in J$, and the function $h_j$ is Dini differentiable at $\bar{x}$ for all $j \in L$, the function $g_i$ is continuous at $\bar{x}$ for all $i \notin I(\bar{x})$. Then, the following system has no solution $v \in T$:

$$f^+_s(\bar{x}; v) < 0,$$

$$f^+_k(\bar{x}; v) \leq 0 \quad (\forall k \in J, k \neq s),$$

$$g_i^-(\bar{x}; v) \leq 0 \quad (\forall i \in I(\bar{x})),$$

$$h'_j(\bar{x}; v) = 0 \quad (\forall j \in L).$$
Assumption 3.1 is fulfilled and the constraint qualification \((CQ)\) Let Theorem 3.3 for local Pareto minimum of Problem (MP). Therefore, there exists a sequence \(t_n \downarrow 0\) such that \(\mathbf{x} + t_nv_0 \in Q_s(\mathbf{v})\), and so \(\mathbf{x} + t_nv_0 \in C\) and
\[
\begin{align*}
  f_k(\mathbf{x} + t_nv_0) &\leq f_k(\mathbf{x}) \quad (\forall k \in J, k \neq s), \\
  g_i(\mathbf{x} + t_nv_0) &\leq 0 \quad (\forall i \in I(\mathbf{x})), \\
  h_j(\mathbf{x} + t_nv_0) &= 0 \quad (\forall j \in L).
\end{align*}
\]
For \(i \notin I(\mathbf{x})\), one has that \(g_i(\mathbf{x}) < 0\). In view of the continuity of \(g_i\) \((i \notin I(\mathbf{x}))\), there is a natural number \(N\) such that for all \(n \geq N\), \(g_i(\mathbf{x} + t_nv_0) \leq 0\) \((\forall i \notin I(\mathbf{x}))\). Hence, for all \(n \geq N\), \(g_i(\mathbf{x} + t_nv_0) \leq 0\) \((\forall i \in I)\). Since \(\mathbf{x}\) is a local Pareto minimum of (MP), it results that for all \(n \geq N\), \(f_s(\mathbf{x} + t_nv_0) \geq f_s(\mathbf{x})\), which leads to the following
\[
\begin{align*}
  f_s^+(\mathbf{x}; v_0) \geq \limsup_{n \to +\infty} t_n (f_s(\mathbf{x} + t_nv_0) - f_s(\mathbf{x})) \geq 0.
\end{align*}
\]
But this conflicts with (2). This completes the proof. \(\square\)

To derive necessary conditions for efficiency of Problem (MP) we introduce the following assumption.

**Assumption 3.1** There exists an index \(s \in J\) such that the function \(f_s\) admits a nonempty bounded upper semi-regular convexificator \(\partial^* f_s(\mathbf{x})\) at \(\mathbf{x}\); for every \(k \in J, k \neq s\), and \(i \in I(\mathbf{x})\), the functions \(f_k\) and \(g_i\) admit upper convexificators \(\partial^* f_k(\mathbf{x})\) and \(\partial^* g_i(\mathbf{x})\) at \(\mathbf{x}\), respectively; all the functions \(g_i(i \notin I(\mathbf{x}))\) are continuous at \(\mathbf{x}\); all the functions \(h_j(j \in L)\) are Gâteaux differentiable at \(\mathbf{x}\) with Gâteaux derivatives \(\nabla G h_j(\mathbf{x})\).

For \(s \in J\) and a nonempty closed convex subcone \(T\) of \(Z(C; \mathbf{x})\), we set
\[
H_T^+(\mathbf{x}) = \bigcup \left\{ \mathrm{co} \partial^* f_s(\mathbf{x}) + \sum_{k \in J, k \neq s} \lambda_k \mathrm{co} \partial^* f_k(\mathbf{x}) + \sum_{i \in I(\mathbf{x})} \mu_i \mathrm{co} \partial^* g_i(\mathbf{x}) + \sum_{j \in L} \gamma_j \nabla G h_j(\mathbf{x}) + T^0 : \lambda_k \geq 0 \ (\forall k \in J, k \neq s), \mu_i \geq 0 \ (\forall i \in I(\mathbf{x})), \gamma_j \in \mathbb{R} \ (\forall j \in L) \right\},
\]
where \(\mathrm{co}\) indicates the convex hull.

We are now in a position to formulate a Karush-Kuhn-Tucker necessary condition for local Pareto minimum of Problem (MP).

**Theorem 3.3** Let \(\mathbf{x}\) be a local Pareto minimum of Problem (MP). Assume that Assumption 3.1 is fulfilled and the constraint qualification \((CQ_s)\) holds for some \(s \in J\). Suppose, in addition, that the set \(H_T^+(\mathbf{x})\) is weak* closed for some nonempty closed convex subcone \(T \subseteq Z(C; \mathbf{x})\) with vertex at the origin. Then, there exist \(\lambda_k \geq 0 \ (\forall k \in J, k \neq s), \overline{\mu}_i \geq 0 \ (\forall i \in I(\mathbf{x})), \overline{\gamma}_j \in \mathbb{R} \ (\forall j \in L)\) such that
\[
\begin{align*}
  0 \in \mathrm{co} \partial^* f_s(\mathbf{x}) + \sum_{k \in J, k \neq s} \lambda_k \mathrm{co} \partial^* f_k(\mathbf{x}) + \sum_{i \in I(\mathbf{x})} \overline{\mu}_i \mathrm{co} \partial^* g_i(\mathbf{x}) + \sum_{j \in L} \overline{\gamma}_j \nabla G h_j(\mathbf{x}) + T^0.
\end{align*}
\]
Proof We invoke Theorem 3.2 to deduce that the system (2) - (5) has no solution \( v \in T \), where (9) is of the form: \( \langle \nabla G h_j(\mathbf{x}), v \rangle = 0 \) (\( \forall j \in L \)). Hence, according to Assumption 3.1, the following system is also impossible \( v \in T \):

\[
\sup_{\xi_s \in \text{co} \partial^* f_s(\mathbf{x})} \langle \xi_s, v \rangle < 0, \\
\sup_{\xi_k \in \text{co} \partial^* f_k(\mathbf{x})} \langle \xi_k, v \rangle \leq 0 \ (\forall k \in J, k \neq s), \\
\sup_{\zeta_i \in \text{co} \partial^* g_i(\mathbf{x})} \langle \zeta_i, v \rangle \leq 0 \ (\forall i \in I(\mathbf{x})), \\
\langle \nabla G h_j(\mathbf{x}), v \rangle = 0 \ (\forall j \in L). 
\]

Let us show that \( 0 \in H^+_T(\mathbf{x}) \).

Assume the contrary, that \( 0 \notin H^+_T(\mathbf{x}) \). Observing that \( H^+_T(\mathbf{x}) \) is convex and weak* closed, a separation theorem for a weak* closed convex set and a point outside that set (see, [5], Theorem 3.4) can be applied, and yields the existence of \( 0 \neq v_0 \in X \) such that

\[
\langle \zeta, v_0 \rangle < 0 \ (\forall \zeta \in H^+_T(\mathbf{x})). 
\]

This implies that

\[
\langle \xi_s, v_0 \rangle + \sum_{k \in J, k \neq s} \lambda_k \langle \xi_k, v_0 \rangle + \sum_{i \in I(\mathbf{x})} \mu_i \langle \zeta_i, v_0 \rangle + \sum_{j \in L} \gamma_j \langle \nabla G h_j(\mathbf{x}), v_0 \rangle + \langle \eta, v_0 \rangle < 0, 
\]

for all \( \xi_s \in \text{co} \partial^* f_s(\mathbf{x}), \lambda_k \geq 0, \xi_k \in \text{co} \partial^* f_k(\mathbf{x})(k \in J, k \neq s), \mu_i \geq 0, \zeta_i \in \text{co} \partial^* g_i(\mathbf{x})(i \in I(\mathbf{x})), \gamma_j \in \mathbb{R}(j \in L), \eta \in T^0 \).

For \( \lambda_k = 0(\forall k \in J, k \neq s), \mu_i = 0(\forall i \in I(\mathbf{x})), \gamma_j = 0(\forall j \in L), \eta = 0 \), in view of the boundness of \( \partial^* f_s(\mathbf{x}) \), it follows from (13) that

\[
\sup_{\xi_s \in \text{co} \partial^* f_s(\mathbf{x})} \langle \xi_s, v_0 \rangle < 0. 
\]

Let us show that

\[
\sup_{\xi_k \in \text{co} \partial^* f_k(\mathbf{x})} \langle \xi_k, v_0 \rangle \leq 0 \ (\forall k \in J, k \neq s). 
\]

If this were not so, there would exist \( k_0 \in J, k_0 \neq s \) such that

\[
\sup_{\xi_k \in \text{co} \partial^* f_{k_0}(\mathbf{x})} \langle \xi_{k_0}, v_0 \rangle > 0. 
\]
Then, for \( \lambda_k = 0(\forall k \in J, k \neq k_0, s), \mu_i = 0(\forall i \in I(\overline{x})), \gamma_j = 0(\forall j \in L), \eta = 0, \xi_s \in \partial^* f_s(\overline{x}), \) by taking \( \lambda_{k_0} \) be large enough, we get that

\[
\langle \xi_s, v_0 \rangle + \lambda_{k_0} \sup_{\xi_{k_0} \in \text{co} \partial^* f_{k_0}(\overline{x})} \langle \xi_{k_0}, v_0 \rangle > 0,
\]

as \( |\langle \xi_s, v_0 \rangle| < +\infty. \) But it follows from (13) that

\[
\langle \xi_s, v_0 \rangle + \lambda_{k_0} \sup_{\xi_{k_0} \in \text{co} \partial^* f_{k_0}(\overline{x})} \langle \xi_{k_0}, v_0 \rangle \leq 0.
\]

Thus we arrive at a contradiction, and so (15) follows. Analogously, we obtain

\[
\sup_{\zeta_i \in \text{co} \partial^* g_i(\overline{x})} \langle \zeta_i, v_0 \rangle \leq 0 (\forall i \in I(\overline{x})).
\]

(16)

Moreover, we also can see that

\[
\langle \nabla G h_j(\overline{x}), v_0 \rangle = 0 (\forall j \in L).
\]

(17)

Indeed, suppose that this were false, that is \( \langle \nabla G h_{j_0}(\overline{x}), v_0 \rangle \neq 0 \) for some \( j_0 \in L. \) Then for \( \lambda_k = 0(\forall k \in J, k \neq s), \mu_i = 0(\forall i \in I(\overline{x})), \eta = 0, \xi_s \in \partial^* f_s(\overline{x}), \) in view of the boundness of \( \xi_s, \) by letting \( \gamma_{j_0} \) be sufficiently large if \( \langle \nabla G h_{j_0}(\overline{x}), v_0 \rangle > 0, \) while \( \gamma_{j_0} < 0 \) with its absolute value be large enough if \( \langle \nabla G h_{j_0}(\overline{x}), v_0 \rangle < 0, \) we shall arrive at a contradiction with (13), and so (17) holds.

It can see that \( v_0 \in T. \) In fact, if it were not so, there would exist \( \eta_0 \in T^0 \) such that \( \langle \eta_0, v_0 \rangle > 0. \) By letting \( \lambda_k = 0(\forall k \in J, k \neq s), \mu_i = 0(\forall i \in I(\overline{x})), \gamma_j = 0(\forall j \in L), \) for \( \alpha \) sufficiently large, \( \alpha \eta_0 \in T^0, \) and so we arrive at a contradiction with (13). Therefore, \( \langle \eta, v_0 \rangle \leq 0(\forall \eta \in T^0). \) Since \( T \) is closed convex, it is also weakly closed, and hence, \( v_0 \in T^0 = T. \) It follows from (14) - (17) that the system (7) - (10) has a solution \( v_0 \in T: \) a contradiction. Consequently, (11) holds, and so there exist \( \lambda_k \geq 0 (\forall k \in J, k \neq s), \pi_i \geq 0 (\forall i \in I(\overline{x})), \tau_j \in \mathbb{R} (\forall j \in L) \) such that the inclusion (6) holds. The proof is complete. \( \square \)

**Remark 3.1** In Theorem 3.3, \( \partial^* f_k(\overline{x})(k \in J, k \neq s) \) and \( \partial^* g_i(\overline{x})(i \in I(\overline{x})) \) may be unbounded. The functions \( f_k(k \in J, k \neq s), g_i(i \in I(\overline{x})), h_j(j \in L) \) may be not continuous.

### 4 Strong Karush-Kuhn-Tucker necessary conditions

To derive strong Karush-Kuhn-Tucker necessary conditions for efficiency of Problem (MP) with positive Lagrange multipliers corresponding to all the components of the objective, we introduce the following assumption.

**Assumption 4.1** For every \( k \in J, \) the function \( f_k \) admits a nonempty bounded upper semi-regular convexificator \( \partial^* f_k(\overline{x}) \) at \( \overline{x}; \) for every \( i \in I(\overline{x}), \) the function \( g_i \)
admits upper convexificator $\partial^* g_i(\bar{x})$ at $\bar{x}$; all the functions $g_i(i \notin I(\bar{x}))$ are continuous at $\bar{x}$; all the functions $h_j(j \in L)$ are Gâteaux differentiable at $\bar{x}$.

In what follows we shall give a strong Karush-Kuhn-Tucker necessary condition for local Pareto minimum in which Lagrange multipliers corresponding to all the components of the objective function are positive.

**Theorem 4.1** Let $\bar{x}$ be a local Pareto minimum of Problem (MP). Assume that Assumption 4.1 is fulfilled, the constraint qualification (CQ$_s$) holds for all $s \in J$, and the set $H^s_T(\bar{x})$ weak* closed for some nonempty closed convex subcone $T$ of $Z(C; \bar{x})$ with vertex at the origin and all $s \in J$. Then, there exist $\bar{\lambda}_k > 0 \ (\forall k \in J), \bar{\mu}_i \geq 0 \ (\forall i \in I(\bar{x})), \bar{\gamma}_j \in \mathbb{R} \ (\forall j \in L)$ such that

$$0 \in \sum_{k \in J} \bar{\lambda}_k \co\partial^* f_k(\bar{x}) + \sum_{i \in I(\bar{x})} \bar{\mu}_i \co\partial^* g_i(\bar{x}) + \sum_{j \in L} \bar{\gamma}_j \nabla G h_j(\bar{x}) + T^0. \quad (18)$$

**Proof** It is easy to see that Assumption 4.1 implies Assumption 3.1 for all $s \in J$. We invoke Theorem 3.3 to deduce that for every $s \in J$, there exist $\lambda_k^{(s)} \geq 0 \ (\forall k \in J, k \neq s), \mu_i^{(s)} \geq 0 \ (\forall i \in I(\bar{x}))$ and $\gamma_j^{(s)} \in \mathbb{R} \ (\forall j \in L)$ such that

$$0 \in \co\partial^* f_s(\bar{x}) + \sum_{k \in J, k \neq s} \lambda_k^{(s)} \co\partial^* f_k(\bar{x})$$

$$+ \sum_{i \in I(\bar{x})} \mu_i^{(s)} \co\partial^* g_i(\bar{x}) + \sum_{j \in L} \gamma_j^{(s)} \nabla G h_j(\bar{x}) + T^0, \quad (18)$$

Taking $s = 1, \ldots, m$ in (18) and adding up both sides of the obtained inclusions, we arrive at

$$0 \in \sum_{k \in J} \bar{\lambda}_k \co\partial^* f_k(\bar{x}) + \sum_{i \in I(\bar{x})} \bar{\mu}_i \co\partial^* g_i(\bar{x}) + \sum_{j \in L} \bar{\gamma}_j \nabla G h_j(\bar{x}) + T^0,$$

where $\bar{\lambda}_k = 1 + \sum_{s \in J, s \neq k} \lambda_k^{(s)} > 0 \ (\forall k \in J), \bar{\mu}_i = \sum_{s \in J} \mu_i^{(s)} \geq 0 \ (\forall i \in I(\bar{x})), \bar{\gamma}_j = \sum_{s \in J} \gamma_j^{(s)} \in \mathbb{R} \ (\forall j \in L)$, as was to be shown.  

**Remark 4.1** In Theorem 4.1, $\partial^* g_i(\bar{x})(i \in I(\bar{x}))$ may be unbounded. The functions $g_i(i \in I(\bar{x})), h_j(j \in L)$ may be not continuous.

**References**


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