HIGHER-ORDER OPTIMALITY CONDITIONS IN MULTIOBJECTIVE OPTIMIZATION

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Abstract In this paper we present higher-order necessary optimality conditions for local weak and Pareto minima of multiobjective optimization problems involving a cone constraint and a set constraint with \( n \)-times Gâteaux differentiable functions in terms of higher-order Gâteaux derivatives and higher-order tangent cones in infinite dimension. Higher-order sufficient optimality conditions are presented for strict local Pareto minima of order \( n \) in finite dimension together with some illustrative examples.

Keywords Higher-order optimality conditions, higher-order tangent cones, local Pareto minimum, strict local Pareto minimum of order \( n \).

1 Introduction

In recent years, the theory of optimality conditions has been extensively developed by many authors. There are many papers dealing with higher-order optimality conditions (see, e.g., [1], [2], [4], [5], [7–11], [13–15], and references therein). Studniarski [15] introduced the notion of isolated local minimum with order \( n \) for constrained optimization problems and derived higher-order necessary and sufficient conditions for isolated local minima. For multiobjective optimization problems, Jiménez [4] introduced the notion of strict local Pareto minimum of order \( n \) and developed a theory on strict Pareto minimum of order \( n \). Constantin [1] and Lee–Pavel [11] established higher-order optimality conditions for constrained optimization problems via higher-order tangent cones. Note that in [11] and [1] higher-order optimality conditions are only established for scalar optimization problems.

The purpose of this paper is to establish higher-order necessary efficiency conditions for multiobjective optimization problems involving cone constraints and a set constraint with \( n \)-times Gâteaux differentiable data via higher-order tangent cones, and higher-order sufficient conditions for strict local Pareto minima of higher-order.

2 Notions and definitions

Let \( D \) be a nonempty subset of a real normed linear space \( X \). Recall [12, 13] that an element \( v \in X \) is called a tangent vector to \( D \) at \( \bar{x} \in \text{cl}D \) iff

\[
\lim_{t \downarrow 0} \frac{1}{t} d(\bar{x} + tv; D) = 0 ,
\]

(1)
where $d(x; D)$ stands for the distance from $x$ to $D$, $\text{cl} D$ is the closure of $D$. The set of all tangent vectors to $D$ at $\vec{x}$ is denoted by $T_{\vec{x}} D$, and is called the tangent cone to $D$ at $x$. Note that $T_{\vec{x}} D$ is a nonempty closed cone containing $0 \in X$. Moreover, (1) is equivalent to the existence of a function $\gamma : (0, +\infty) \to X$ with $\gamma(t) \to 0$ as $t \downarrow 0$, and

$$\vec{x} + t(v + \gamma(t)) \in D \quad (\forall \, t > 0).$$

(2)

For a given natural number $n \geq 2$, following [13], an element $v \in X$ is said to be an $n$th order tangent vector to $D$ at $\vec{x}$ iff there exist $v_i \in X \ (i = 1, \ldots, n - 1)$ such that

$$\lim_{t \downarrow 0} \frac{1}{t^n} d(\vec{x} + tv_1 + \frac{t^2}{2!}v_2 + \cdots + \frac{t^n}{n!}v_n; D) = 0.$$  

(3)

The vectors $v_1, \ldots, v_{n-1}$ satisfying (3) are said to be associated with $v_n$ (or the corresponding vectors of $v_n$). Denote by $T_{\vec{x}}^n D$ the set of all $n$th order tangent vectors to $D$ at $\vec{x}$, where $T_{\vec{x}}^n D = T_{\vec{x}} D$. It should be noted here that (see [1]) $v_n \in T_{\vec{x}}^n D$ with the associated vectors $v_1, \ldots, v_{n-1}$ is equivalent to the existence of a function $\gamma_n : (0, +\infty) \to X$ with $\gamma_n(t) \to 0$ as $t \downarrow 0$, and

$$\vec{x} + tv_1 + \frac{t^2}{2!}v_2 + \cdots + \frac{t^n}{n!}(v_n + \gamma_n(t)) \in D \quad (\forall \, t > 0).$$

(4)

Moreover, $v_n \in T_{\vec{x}}^n D$ implies that $v_i \in T_{\vec{x}}^k D, i = 1, \ldots, n - 1$ (see [1]). For such construction, $T_{\vec{x}}^n D$ is a cone (see [8]).

Let us denote $(v)^k = (v, \ldots, v) \in X^k$. Let $f$ be a mapping from $X$ into a real normed linear space $Y$. Recall that $f$ is Gâteaux differentiable at $\vec{x}$ iff there exists a continuous linear mapping $A_1$ from $X$ into $Y$ such that

$$f(\vec{x} + tv) = f(\vec{x}) + tA_1(v) + o(t) \ (\forall \, v \in X),$$

where $\|o(t)\| / |t| \to 0$ as $t \to 0$. The mapping $A_1$ is said to be the Gâteaux derivative of $f$ at $\vec{x}$ and is denoted by $f'_G(\vec{x})$. The mapping $f$ is $n$-times Gâteaux differentiable at $\vec{x} \ (n \geq 2)$ iff $f$ is Gâteaux differentiable at $\vec{x}$ and there exist continuous multilinear symmetric mappings $A_k$ from $X^k$ into $Y$ (continuous linear symmetric in $k$ variables), $k = 2, \ldots, n$, such that

$$f(\vec{x} + tv) = f(\vec{x}) + tA_1(v) + \frac{t^2}{2!}A_2(v)^2 + \cdots + \frac{t^n}{n!}A_n(v)^n + o(t^n) \ (\forall \, v \in X),$$

where $A_1 = f'_G(\vec{x}), \ |o(t^n)|/|t|^n \to 0$, as $t \to 0$. Note that symmetric means it does not change under permutation of variables. For the correctness of this definition, the symmetric multilinear mapping $A_n$ should be uniquely determined by the respective form $v \to A_n(v)^n$ (see, e.g., [6]). The continuous multilinear symmetric mapping $A_k$ is the $k$th order Gâteaux derivative of $f$ at $\vec{x}$ and is denoted by $f^{(k)}_G(\vec{x})$. Thus for a function $f$ which is $n$-times Gâteaux differentiable at $\vec{x}$, $f$ can be expanded as

$$f(\vec{x} + tv) = f(\vec{x}) + tf'_G(\vec{x})(v) + \frac{t^2}{2!}f^{(2)}_G(\vec{x})(v)^2 + \cdots + \frac{t^n}{n!}f^{(n)}_G(\vec{x})(v)^n + o(t^n) \ (\forall \, v \in X).$$

(5)
In case \( f \) is \( n \)-times Fréchet differentiable at \( \bar{x} \), we have the following Taylor expansion:

\[
f(\bar{x} + v) = f(\bar{x}) + f'(\bar{x})(v) + \frac{1}{2!} f^{(2)}(\bar{x})(v)^2 + \cdots + \frac{1}{n!} f^{(n)}(\bar{x})(v)^n + r(v),
\]
where \( f'(\bar{x}) \) is the first order Fréchet derivative of \( f \) at \( \bar{x} \), \( f^{(k)}(\bar{x}) \) is \( k \)th order Fréchet derivative of \( f \) at \( \bar{x} \) \((k \geq 2)\), \( ||r(v)||/||v||^n \to 0 \), as \( v \to 0 \). In this case, \( f^{(k)}(\bar{x}) = f^{(k)}_G(\bar{x}) \) \((k = 1, \ldots, n)\).

### 3 Higher-order necessary optimality conditions

Let \( f \) and \( g \) be mappings from a real linear normed space \( X \) into other real linear normed spaces \( Y \) and \( Z \), respectively. Let \( C \) be a subset of \( X \) and \( S \) be a closed convex cone in \( Z \). In this section, we shall be concerned with the following multiobjective optimization problem (MP1):

\[
\min f(x) \text{ s.t. } x \in M_1 := \left\{ x \in C : -g(x) \in S \right\}.
\]

Recall that a point \( \bar{x} \in M_1 \) is said to be a local weak minimum (resp. local Pareto minimum) of Problem (MP1) iff there exists a number \( \delta > 0 \) such that

\[
(f(x) - f(\bar{x}) \notin -\text{int}Q \quad (\forall x \in M_1 \cap B(\bar{x}; \delta))
\]

(\( f(x) - f(\bar{x}) \notin Q \backslash \{0\}, \quad \forall x \in M_1 \cap B(\bar{x}; \delta) \)),

where \( B(\bar{x}; \delta) \) stands for the open ball of radius \( \delta \) around \( \bar{x} \), \( \text{int}Q \) indicates the interior of \( Q \). Note that with local weak minima we suppose that \( \text{int}Q \neq \emptyset \). Following [4], the point \( \bar{x} \) is called a strict local Pareto minimum of order \( n \) of Problem (MP1) iff there exist numbers \( \delta > 0 \) and \( \alpha > 0 \) such that

\[
(f(x) + Q) \cap B(f(\bar{x}); \alpha||x - \bar{x}||^n) = \emptyset \quad (\forall x \in M_1 \cap B(\bar{x}; \delta) \backslash \{\bar{x}\}).
\]

Now given a function \( f \) which is \( m \)-times Gâteaux differentiable at \( \bar{x} \) and \( m \) vectors \( v_1, v_2, \ldots, v_m \in X \), \( m \) positive integer, we denote by \( H^f_m(\bar{x}; v_1, \ldots, v_m) \) the expression below inspired by a similar expression defined in [1] for a function of class \( C^m \).

\[
H^f_m(\bar{x}; v_1, \ldots, v_m) = \sum_{k=1}^{m} \frac{m!}{k!} \left[ \sum_{i_1, \ldots, i_k \in \{1, \ldots, m\}} \frac{1}{i_1! \cdots i_k!} f^{(k)}_G(\bar{x})(v_{i_1}, \ldots, v_{i_k}) \right],
\]

where \( \{1, m\} \) indicates the set \( \{1, \ldots, m\} \). For \( m = 1, 2, 3 \), one has

\[
\begin{align*}
H^f_1(\bar{x}; v_1) &= f'_G(\bar{x})(v_1), \\
H^f_2(\bar{x}; v_1, v_2) &= f'_G(\bar{x})(v_2) + f^{(2)}_G(\bar{x})(v_1)^2, \\
H^f_3(\bar{x}; v_1, v_2, v_3) &= f'_G(\bar{x})(v_3) + 3f^{(2)}_G(\bar{x})(v_1, v_2) + f^{(3)}_G(\bar{x})(v_1, v_2)^3.
\end{align*}
\]

We shall begin with a higher-order necessary condition for local weak minima.
Theorem 3.1 Let \( \bar{x} \) be a local weak minimum of Problem (MP1) and \( \text{int} S \neq \emptyset \). Assume that \( f \) (resp. \( g \)) is \( n \)-times (resp. \( m \)-times) Gâteaux differentiable at \( \bar{x} \) (\( m \leq n \)). Then, for every \( v_n \in T_{\bar{x}}^n C \) with the associated vectors \( v_1, \ldots, v_{n-1} \) satisfying

\[
H^0_i(\bar{x}; v_1, \ldots, v_k) \in -S, \quad k = 1, \ldots, m - 1,
\]

\[
H^0_m(\bar{x}; v_1, \ldots, v_m) \in -\text{int} S.
\]

\[
H^0_j(\bar{x}; v_1, \ldots, v_j) = 0, \quad j = 1, \ldots, n - 1,
\]

we have

\[
H^0_n(\bar{x}; v_1, \ldots, v_n) \notin -\text{int} Q.
\]

Proof Since \( \bar{x} \) is a local weak minimum of (MP1), there exists a number \( \delta > 0 \) such that

\[
f(x) - f(\bar{x}) \in -(Y \setminus \text{int} Q) \quad (\forall x \in M_1 \cap B(\bar{x}; \delta)).
\]

Observe that for \( v_n \in T_{\bar{x}}^n C \) with the associated vectors \( v_i \in T_{\bar{x}}^i C \) (\( i = 1, \ldots, n - 1 \)), we have

\[
\lim_{t \downarrow 0} \frac{1}{t^n} d(\bar{x} + tv_1 + \frac{t^2}{2!} v_2 + \cdots + \frac{t^n}{n!} v_n; C) = 0.
\]

This is equivalent to the existence of a function \( \gamma_n : (0, +\infty) \to X \) with \( \gamma_n(t) \to 0 \) as \( t \downarrow 0 \), and

\[
\bar{x} + tv_1 + \frac{t^2}{2!} v_2 + \cdots + \frac{t^n}{n!} (v_n + \gamma_n(t)) \in C \quad (\forall t > 0).
\]

Putting \( \varphi(t) = v_1 + \frac{t^2}{2!} v_2 + \cdots + \frac{t^{n-1}}{(n-1)!} (v_n + \gamma_n(t)) \), it follows from (11) that for all \( t > 0 \) small enough,

\[
\bar{x} + t\varphi(t) \in C \cap B(\bar{x}; \delta).
\]

Since \( g \) is \( m \)-times Gâteaux differentiable at \( \bar{x} \) (\( m \leq n \)), it can be expanded as follows

\[
g(\bar{x} + tv_1 + \frac{t^2}{2!} v_2 + \cdots + \frac{t^n}{n!} (v_n + \gamma_n(t))) = g(\bar{x}) + t g'_G(\bar{x}) (\varphi(t)) +
\]

\[
+ \frac{t^2}{2!} g^{(2)}_G (\bar{x}) (\varphi(t))^2 + \cdots + \frac{t^m}{m!} g^{(m)}_G (\bar{x}) (\varphi(t))^m + o(t^m)
\]

\[
= g(\bar{x}) + t g'_G(\bar{x}) (v_1) + \frac{t^2}{2!} \left[ g^{(2)}_G (\bar{x}) (v_1)^2 + g'_G(\bar{x}) (v_2) \right] +
\]

\[
+ \frac{t^3}{3!} \left[ g^{(3)}_G (\bar{x}) (v_1)^3 + 3 g^{(2)}_G (\bar{x}) (v_1 v_2) + g'_G(\bar{x}) (v_3) \right] +
\]

\[
\cdots + \frac{t^m}{m!} H^0_m(\bar{x}; v_1, \ldots, v_m) + o(t^m),
\]

Moreover, by (7) it results that for all \( t > 0 \) small enough,

\[
H^0_m(\bar{x}; v_1, \ldots, v_m) + \frac{o(t^m)}{t^m} \in -S.
\]

Combining (6), (13) and (14) yields that for all \( t > 0 \) small enough,

\[
g(\bar{x} + t\varphi(t)) \in -S.
\]
Due to (12) and (15) it holds that $\bar{x} + t \varphi(t) \in M_1 \cap B(\bar{x}; \delta)$ for all $t > 0$ small enough. Consequently, it follows from (10) that for all $t > 0$ small enough,

$$f(\bar{x} + t \varphi(t)) - f(\bar{x}) \in - (Y \setminus \text{int} Q). \quad (16)$$

On the other hand, in view of the $n$-times Gâteaux differentiability of $f$ at $\bar{x}$, Taylor’s expansion of $f$ at $\bar{x}$ can be written as

$$f(\bar{x} + t \varphi(t)) - f(\bar{x}) = t f'_G(\bar{x})(\varphi(t)) + \frac{t^2}{2!} f''_G(\bar{x})(\varphi(t))^2 + \frac{t^3}{3!} f^{(3)}_G(\bar{x})(\varphi(t))^3 + \cdots + \frac{t^n}{n!} f^{(n)}_G(\bar{x})(\varphi(t))^n + o(t^n)$$

which implies that

$$0 \leq \frac{t^n}{n!} H^f_n(\bar{x}; v_1, \ldots, v_n) + o(t^n),$$

Combining (8), (16) and (17) yields that for all $t > 0$ small enough,

$$\frac{t^n}{n!} H^f_n(\bar{x}; v_1, \ldots, v_n) + o(t^n) \in - (Y \setminus \text{int} Q),$$

which implies that

$$H^f_n(\bar{x}; v_1, \ldots, v_n) \notin - \text{int} Q,$$

as was to be shown. \Box

Let us consider the case $Y = \mathbb{R}^r$, $Q = \mathbb{R}^r_+$, and so $f = (f_1, \ldots, f_r)$. A higher-order necessary condition for local Pareto minimality can be stated as follows.

**Theorem 3.2** Let $\bar{x}$ be a local Pareto minimum of Problem (MP1). Assume that $f_1, \ldots, f_r$ (resp. $g$) are $n$-times (resp. $m$-times) Gâteaux differentiable at $\bar{x}$ ($m \leq n$).

Then, for any $s = 1, \ldots, n$, $v_n \in T^*_x C$ with associated vectors $v_1, \ldots, v_{n-1}$ satisfying

$$H^g_k(\bar{x}; v_1, \ldots, v_k) \in - S, \quad k = 1, \ldots, m - 1,$$

$$H^g_m(\bar{x}; v_1, \ldots, v_m) \in - \text{int} S,$$

$$H^l_i(\bar{x}; v_1, \ldots, v_i) \leq 0 \quad (i = 1, \ldots, n - 1; \ k \neq s),$$

$$H^l_s(\bar{x}; v_1, \ldots, v_n) < 0, \quad (k \neq s),$$

$$H^l_j(\bar{x}; v_1, \ldots, v_j) = 0, \quad j = 1, \ldots, n - 1,$$

we have

$$H^l_n(\bar{x}; v_1, \ldots, v_n) \geq 0.$$

**Proof** Since $\bar{x}$ is a local Pareto minimum of (MP1), for $s \in \{1, r\}$, $\bar{x}$ is a local minimum of the following scalar optimization problem (P):

$$\min f_s(x) \text{ s.t. } x \in M := \{x \in C : f_k(x) \leq f_k(\bar{x}) \ (k = 1, \ldots, r; \ k \neq s), - g(x) \in S\}.$$
Then there exists a neighborhood $U$ of $\bar{x}$ such that

$$f_s(x) \geq f_s(\bar{x}) \quad (\forall x \in M \cap U). \quad (18)$$

For $v_n \in T_{\bar{x}}C$, there exists a function $\gamma_n : (0, +\infty) \to X$ with $\gamma_n(t) \to 0$ as $t \downarrow 0$, and for all $t > 0$,

$$\bar{x} + tv_1 + \frac{t^2}{2!} v_2 + \cdots + \frac{t^n}{n!} (v_n + \gamma_n(t)) \in C.$$  

We set $\varphi(t) = v_1 + \frac{t}{1!} v_2 + \cdots + \frac{t^{n-1}}{(n-1)!} (v_n + \gamma_n(t))$ and get $\bar{x} + t \varphi(t) \in C$. Hence, for all $t > 0$ small enough,

$$\bar{x} + t \varphi(t) \in C \cap U. \quad (19)$$

By an argument analogous to that used in the proof of Theorem 3.1 we obtain that for all $t > 0$ small enough,

$$f_k(\bar{x} + t \varphi(t)) \leq f_k(\bar{x}) \quad (\forall k = 1, \ldots, r; \ k \neq s), \quad (20)$$

$$-g(\bar{x} + t \varphi(t)) \in S. \quad (21)$$

Combining (19)–(21) yields that $\bar{x} + t \varphi(t) \in M \cap U$ for all $t > 0$ small enough. Hence, it follows from (18) that for all $t > 0$ small enough,

$$f_s(\bar{x} + t \varphi(t)) \geq f_s(\bar{x}).$$

In view of the Gâteaux differentiability of $f_s$ and in the same way as in the proof of Theorem 3.1 we get that for all $t > 0$ small enough,

$$f_s(\bar{x} + t \varphi(t)) - f_s(\bar{x}) = \frac{t^n}{n!} H_n^{f_s}(\bar{x}; v_1, \ldots, v_n) + o(t^n) \geq 0,$$

which implies that

$$H_n^{f_s}(\bar{x}; v_1, \ldots, v_n) \geq 0,$$

which completes the proof.

### 4 Higher-order sufficient optimality conditions

This section deals with higher-order sufficient conditions for strict local Pareto minima of order $n$ of Problem (MP2):

$$\min f(x) \text{ s.t. } x \in M_2 := \left\{ x \in C : -g(x) \in S, h(x) = 0 \right\},$$

where $f, g, S, C$ are as in Problems (MP1) with $\text{int}S \neq \emptyset$, $h$ is a mapping from $X$ into a real linear normed space $W$, and $\text{dim}X < +\infty$. Denote by $S$ the unit sphere in $X$. We set $S_{g(\bar{x})} = \text{cl}(\text{cone}(S + g(\bar{x})))$, in which $\text{cl}(\text{cone}(S + g(\bar{x})))$ is the closure of the convex cone generated by $S + g(\bar{x})$. Note that $\text{int}S_{g(\bar{x})} \neq \emptyset$, as $\text{int}S \neq \emptyset$.

**Theorem 4.1** Let $\bar{x}$ be a feasible point of (MP2). Assume that $C$ is convex, $g$ and $h$ are Gâteaux differentiable at $\bar{x}$, $f$ is $n$-times Gâteaux differentiable at $\bar{x}$. Suppose, furthermore, that the following two conditions hold:
(i) $f^{(i)}_G(\pi)(v)^i \in Q \ (\forall v \in (T_\pi C) \cap S, \ i = 1, \ldots, n - 1)$;

(ii) $f^{(n)}_G(\pi)(v)^n \notin -Q \ (\forall v \in (T_\pi C) \cap S \cap$

\[ \cap \{u : g'_G(\pi)(u) \in -S_{g(\pi)}, \ h'_G(\pi)(u) = 0\}. \]

Then $\pi$ is a strict local Pareto minimum of order $n$ of Problem (MP2).

Proof. Contrary to the conclusion, we suppose that $\pi$ is not a strict local Pareto minimum of order $n$ of (MP2). By Proposition 3.4[4], this assumption yields the existence of $x_m \in M_2$, $x_m \neq \pi$, $x_m \to 0$ and $b_m \in Q$ such that

\[ \lim_{m \to +\infty} \frac{f(x_m) - f(\pi) + b_m}{\|x_m - \pi\|^n} = 0. \]  

(22)

Since $C$ is convex, it holds that $x_m - \pi \in T_\pi C$. Hence, $(x_m - \pi)/t_m \in T_\pi C$, where $t_m = \|x_m - \pi\|$. In view of the compactness of $S$, without loss of generality, we can assume that $v_m := (x_m - \pi)/t_m \to v_0$ with $\|v_0\| = 1$, and so $v_0 \in (T_\pi C) \cap S$, as $T_\pi C$ is closed.

Since $g$ is Gâteaux differentiable at $\pi$, it can be written as

\[ g(x_m) = g(\pi) + t_m g'_G(\pi)(v_m) + o(t_m), \]

where $o(t_m)/t_m \to 0$ as $m \to +\infty$. Hence,

\[ g'_G(\pi)(v_m) + \frac{o(t_m)}{t_m} \in -\text{cone}(S + g(\pi)). \]

By letting $m \to +\infty$, one gets

\[ g'_G(\pi)(v_0) \in -S_{g(\pi)}. \]  

(23)

Since $h$ is Gâteaux differentiable at $\pi$, it can be written as

\[ h(x_m) = h(\pi) + t_m h'_G(\pi)(v_m) + o(t_m), \]

which yields that

\[ h'_G(\pi)(v_0) = 0. \]  

(24)

It follows from (23), (24) and assumption (ii) that

\[ f^{(n)}_G(\pi)(v_0)^n \notin -Q. \]  

(25)

On the other hand, since $f$ is $n$-times Gâteaux differentiable at $\pi$, it can be written as

\[ f(x_m) = f(\pi) + t_m f'_G(\pi)(v_m) + \frac{t^2_m}{2!} f^{(2)}_G(\pi)(v_m)^2 + \cdots + \frac{t^n_m}{n!} f^{(n)}_G(\pi)(v_m)^n + o(t^n_m), \]

where $o(t^n_m)/t^n_m \to 0$ as $m \to +\infty$. Therefore,

\[ f^{(n)}_G(\pi)(v_0)^n = \lim_{m \to +\infty} f^{(n)}_G(\pi)(v_m)^n = \lim_{m \to +\infty} \frac{n!}{t^n_m} \left[ f(x_m) - f(\pi) - \sum_{j=1}^{n-1} \frac{t^j_m}{j!} f^{(j)}_G(\pi)(v_m)_j \right]. \]
The existence of \( f^{(j)}_G(\bar{x})(v_m)^j \) \((j = 1, \ldots, n)\) together with (22) yields the existence of the limit \( \lim_{m \to +\infty} \frac{1}{t_m} [b_m + \sum_{j=1}^{n-1} \frac{t_m}{j!} f^{(j)}_G(\bar{x})(v_m)^j] \). Consequently,

\[
f^{(n)}_G(\bar{x})(v_0)^n = \lim_{m \to +\infty} \frac{n!}{t_m} [f(x_m) - f(\bar{x}) + b_m] - \lim_{m \to +\infty} \frac{n!}{t_m} [b_m + \sum_{j=1}^{n-1} \frac{t_m}{j!} f^{(j)}_G(\bar{x})(v_m)^j]
\]

\[
= - \lim_{m \to +\infty} \frac{n!}{t_m} [b_m + \sum_{j=1}^{n-1} \frac{t_m}{j!} f^{(j)}_G(\bar{x})(v_m)^j].
\]

It can be seen that \( \lim_{m \to +\infty} \frac{1}{t_m} [b_m + \sum_{j=1}^{n-1} \frac{t_m}{j!} f^{(j)}_G(\bar{x})(v_m)^j] \in Q \), as \( b_m \in Q \), \( f^{(j)}_G(\bar{x})(v_m)^j \in Q \) \((i = 1, \ldots, n-1)\) and \( Q \) is closed. Hence, in view of (22) it results that

\[
f^{(n)}_G(\bar{x})(v_0)^n \in -Q.
\]

But this conflicts with (25). The proof is complete.

Let us consider the case \( Y = \mathbb{R}^r \), \( Q = \mathbb{R}^r_+ \) and \( f = (f_1, \ldots, f_r) \). Denote by \( f^{(i)}_{k,G}(\bar{x}) \) the \( i \)th order Gâteaux derivative of \( f_k \) at \( \bar{x} \). A higher-order sufficient condition for strict local Pareto minimum of higher-order can be stated as follows.

**Theorem 4.2** Let \( \bar{x} \) be a feasible point of (MP2). Assume that \( C \) is convex, \( g \) and \( h \) are Gâteaux differentiable at \( \bar{x} \), \( f_s \) is \( n \) times Gâteaux differentiable at \( \bar{x} \) for some \( s \in \{1, n\} \). Suppose also that the following two conditions hold:

(i) \( f^{(i)}_{s,G}(\bar{x})(v)^i \geq 0 \) \((\forall v \in (T_{\bar{x}}C) \cap S, i = 1, \ldots, n-1)\);

(ii) \( f^{(n)}_{s,G}(\bar{x})(v)^n > 0 \) \((\forall v \in (T_{\bar{x}}C) \cap S \cap \cap \{u : g'_{G}(\bar{x})(u) \in -S_{g(\bar{x})}, h'_{G}(\bar{x})(u) = 0\})\).

Then \( \bar{x} \) is a strict local Pareto minimum of order \( n \) of Problem (MP2).

**Proof** Let us consider the following scalar problem (P1):

\[
\min f_s(x) \text{ s.t. } x \in M_2 := \{ x \in C : -g(x) \in S, h(x) = 0 \},
\]

where \( f, g, h, C, S \) are as in Problem (MP2). We shall begin with showing that \( \bar{x} \) is a strict local minimum of order \( n \) for (P1). Suppose by contradiction that \( \bar{x} \) is not a strict local minimum of order \( n \) for (P1). Then there would exist a sequence \( \{x_m\} \subset (M_2) \) with \( x_m \to \bar{x} \) such that

\[
f_s(x_m) < f_s(\bar{x}) + \frac{1}{m} \|x_m - \bar{x}\|^n \quad (\forall m).
\]

Observe that \( x_m - \bar{x} \in T_{\bar{x}}C \), as \( C \) is convex. We set \( t_m = \|x_m - \bar{x}\| \) and get that \( (x_m - \bar{x})/t_m \in T_{\bar{x}}C \). As also in the proof of Theorem 4.1 we can assume that \( v_m := (x_m - \bar{x})/t_m \to v_0 \) with \( \|v_0\| = 1 \), which means that \( v_0 \in S \), and so \( v_0 \in (T_{\bar{x}}C) \cap S \), as \( T_{\bar{x}}C \) is closed.

By an argument analogous to that used for the proof of Theorem 4.1, one gets

\[
g'_{G}(\bar{x})(v_0) \in -S,
\]

\[
h'_{G}(\bar{x})(v_0) = 0.
\]
It follows from (27), (28) and condition (ii) that
\[ f_{s,G}^{(n)}(\pi)(v_0)^n > 0. \]  
(29)

On the other hand, since \( f_s \) is \( n \)-times Gâteaux differentiable at \( \pi \), it can be written as
\[ f_s(x_m) = f_s(\pi) + t_m f'_{s,G}(\pi)(v_m) + \frac{t_m^2}{2!} f''_{s,G}(\pi)(v_m)^2 + \cdots + \frac{t_m^n}{n!} f^{(n)}_{s,G}(\pi)(v_m)^n + o(t_m^n). \]  
(30)

One has that \( f_{s,G}^{(i)}(\pi)(v_m)^i \geq 0 \) \( (i = 1, \ldots, n - 1) \), as \( v_m \in T_\pi C \cap S \). Hence, combining (26) and (30) yields that
\[ \frac{t_m^n}{n!} f^{(n)}_{s,G}(\pi)(v_m)^n + o(t_m^n) \leq f_s(x_m) - f_s(\pi) < \frac{1}{m} \| x_m - \pi \|^n = \frac{t_m^n}{m} \| v_m \|^n. \]

By letting \( m \to +\infty \) we obtain
\[ f_{s,G}^{(n)}(\pi)(v_0)^n \leq 0, \]
which is in contradiction to (29). Hence, \( \pi \) is a strict local minimum of order \( n \) of Problem (P1). It is easy to show that \( \pi \) is a strict local Pareto minimum of order \( n \) of Problem (MP2). The proof is complete.

We close this paper with an example, which illustrates Theorem 4.2.

**Example 4.1** Let \( X = Y = Z = \mathbb{R}^2, S = \mathbb{R}^2_+ \), \( C = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 \leq 1, 0 \leq x_2 \leq 10x_1\}, \pi = (0, 0) \). Let \( f \) and \( g \) be defined on \( \mathbb{R}^2 \) as
\[
\begin{align*}
    f(x) &= (f_1(x), f_2(x)), \\
    f_1(x) &= x_1^n, \text{ where } n \text{ is a natural number } (n \geq 1), \\
    f_2(x) &= -\max\{|x_1|, |x_2|\} \quad (x = (x_1, x_2) \in \mathbb{R}^2), \\
    g(x) &= (g_1(x), g_2(x)), \\
    g_1(x) &= \begin{cases} 
    -x_1 - 1, & \text{if } x_2 = x_1^2, \\
    0, & \text{if otherwise}, 
    \end{cases} \\
    g_2(x) &= \begin{cases} 
    (x_1^2 + x_2^2)\sin\frac{1}{x_1^2 + 2x_2} - (x_1^2 + x_2^2), & \text{if } x_1^2 + x_2^2 \neq 0, \\
    0, & \text{if } x_1^2 + x_2^2 = 0, 
    \end{cases}
\end{align*}
\]

Then \( \pi \) is a feasible point of the following multiobjective optimization problem:
\[
\min f(x), \quad \text{s.t.} \quad -g(x) \in \mathbb{R}^2_+ \quad \text{and} \quad x \in C.
\]

Note that the feasible set \( M_2 = C, T_\pi C = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_2 \leq 10x_1\}, S \) is the sphere in \( \mathbb{R}^2 \). The function \( g_1 \) is Gâteaux differentiable at \( \pi \) with \( g'_1(\pi) = (0, 0) \) \( (g_1 \) is not continuous at \( \pi \). The function \( g_2 \) is Fréchet differentiable at \( \pi \) with \( g_2'(\pi) = (0, 0) \). The function \( f_1 \) is \( n \)-times Fréchet differentiable at \( \pi \) with
\[
\begin{align*}
    f_1^{(i)}(\pi)(v)^i &= 0 \quad (\forall v \in (T_\pi C) \cap S; \quad i = 1, \ldots, n - 1), \\
    f_1^{(n)}(\pi)(v)^n &> 0 \quad (\forall v \in (T_\pi C) \cap S) \cap \{v : \quad g'_2(\pi)(v) \in -S_{g(\pi)}\}).
\end{align*}
\]

Then all the assumptions of Theorem 4.2 are fulfilled and \( \pi \) is a strict local Pareto minimum of order \( n \) of this problem.
References


