Introduction to Real Analysis
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INTRODUCTION TO REAL ANALYSIS
Fourth Edition

Robert G. Bartle
Donald R. Sherbert

University of Illinois, Urbana-Champaign

John Wiley & Sons, Inc.
A TRIBUTE

This edition is dedicated to the memory of Robert G. Bartle, a wonderful friend and colleague of forty years. It has been an immense honor and pleasure to be Bob’s coauthor on the previous editions of this book. I greatly miss his knowledge, his insights, and especially his humor.

November 20, 2010
Urbana, Illinois

Donald R. Sherbert
To Jan, with thanks and love.
The study of real analysis is indispensable for a prospective graduate student of pure or applied mathematics. It also has great value for any student who wishes to go beyond the routine manipulations of formulas because it develops the ability to think deductively, analyze mathematical situations and extend ideas to new contexts. Mathematics has become valuable in many areas, including economics and management science as well as the physical sciences, engineering, and computer science. This book was written to provide an accessible, reasonably paced treatment of the basic concepts and techniques of real analysis for students in these areas. While students will find this book challenging, experience has demonstrated that serious students are fully capable of mastering the material.

The first three editions were very well received and this edition maintains the same spirit and user-friendly approach as earlier editions. Every section has been examined. Some sections have been revised, new examples and exercises have been added, and a new section on the Darboux approach to the integral has been added to Chapter 7. There is more material than can be covered in a semester and instructors will need to make selections and perhaps use certain topics as honors or extra credit projects.

To provide some help for students in analyzing proofs of theorems, there is an appendix on “Logic and Proofs” that discusses topics such as implications, negations, contrapositives, and different types of proofs. However, it is a more useful experience to learn how to construct proofs by first watching and then doing than by reading about techniques of proof.

Results and proofs are given at a medium level of generality. For instance, continuous functions on closed, bounded intervals are studied in detail, but the proofs can be readily adapted to a more general situation. This approach is used to advantage in Chapter 11 where topological concepts are discussed. There are a large number of examples to illustrate the concepts, and extensive lists of exercises to challenge students and to aid them in understanding the significance of the theorems.

Chapter 1 has a brief summary of the notions and notations for sets and functions that will be used. A discussion of Mathematical Induction is given, since inductive proofs arise frequently. There is also a section on finite, countable and infinite sets. This chapter can be used to provide some practice in proofs, or covered quickly, or used as background material and returning later as necessary.

Chapter 2 presents the properties of the real number system. The first two sections deal with Algebraic and Order properties, and the crucial Completeness Property is given in Section 2.3 as the Supremum Property. Its ramifications are discussed throughout the remainder of the chapter.

In Chapter 3, a thorough treatment of sequences is given, along with the associated limit concepts. The material is of the greatest importance. Students find it rather natural though it takes time for them to become accustomed to the use of epsilon. A brief introduction to Infinite Series is given in Section 3.7, with more advanced material presented in Chapter 9.
Chapter 4 on limits of functions and Chapter 5 on continuous functions constitute the heart of the book. The discussion of limits and continuity relies heavily on the use of sequences, and the closely parallel approach of these chapters reinforces the understanding of these essential topics. The fundamental properties of continuous functions on intervals are discussed in Sections 5.3 and 5.4. The notion of a gauge is introduced in Section 5.5 and used to give alternate proofs of these theorems. Monotone functions are discussed in Section 5.6.

The basic theory of the derivative is given in the first part of Chapter 6. This material is standard, except a result of Carathéodory is used to give simpler proofs of the Chain Rule and the Inversion Theorem. The remainder of the chapter consists of applications of the Mean Value Theorem and may be explored as time permits.

In Chapter 7, the Riemann integral is defined in Section 7.1 as a limit of Riemann sums. This has the advantage that it is consistent with the students’ first exposure to the integral in calculus, and since it is not dependent on order properties, it permits immediate generalization to complex- and vector-values functions that students may encounter in later courses. It is also consistent with the generalized Riemann integral that is discussed in Chapter 10. Sections 7.2 and 7.3 develop properties of the integral and establish the Fundamental Theorem of Calculus. The new Section 7.4, added in response to requests from a number of instructors, develops the Darboux approach to the integral in terms of upper and lower integrals, and the connection between the two definitions of the integral is established. Section 7.5 gives a brief discussion of numerical methods of calculating the integral of continuous functions.

Sequences of functions and uniform convergence are discussed in the first two sections of Chapter 8, and the basic transcendental functions are put on a firm foundation in Sections 8.3 and 8.4. Chapter 9 completes the discussion of infinite series that was begun in Section 3.7. Chapters 8 and 9 are intrinsically important, and they also show how the material in the earlier chapters can be applied.

Chapter 10 is a presentation of the generalized Riemann integral (sometimes called the “Henstock-Kurzweil” or the “gauge” integral). It will be new to many readers and they will be amazed that such an apparently minor modification of the definition of the Riemann integral can lead to an integral that is more general than the Lebesgue integral. This relatively new approach to integration theory is both accessible and exciting to anyone who has studied the basic Riemann integral.

Chapter 11 deals with topological concepts. Earlier theorems and proofs are extended to a more abstract setting. For example, the concept of compactness is given proper emphasis and metric spaces are introduced. This chapter will be useful to students continuing on to graduate courses in mathematics.

There are lengthy lists of exercises, some easy and some challenging, and “hints” to many of them are provided to help students get started or to check their answers. More complete solutions of almost every exercise are given in a separate Instructor’s Manual, which is available to teachers upon request to the publisher.

It is a satisfying experience to see how the mathematical maturity of the students increases as they gradually learn to work comfortably with concepts that initially seemed so mysterious. But there is no doubt that a lot of hard work is required on the part of both the students and the teachers.

Brief biographical sketches of some famous mathematicians are included to enrich the historical perspective of the book. Thanks go to Dr. Patrick Muldowney for his photograph of Professors Henstock and Kurzweil, and to John Wiley & Sons for obtaining portraits of the other mathematicians.
Many helpful comments have been received from colleagues who have taught from earlier editions of this book and their remarks and suggestions have been appreciated. I wish to thank them and express the hope that they find this new edition even more helpful than the earlier ones.

November 20, 2010
Urbana, Illinois

Donald R. Sherbert

THE GREEK ALPHABET

A $\alpha$ Alpha  N $\nu$ Nu
B $\beta$ Beta $\Xi$ $\xi$ Xi
$\Gamma$ $\gamma$ Gamma  O $\omicron$ Omicron
$\Delta$ $\delta$ Delta $\Pi$ $\pi$ Pi
$\Epsilon$ $\epsilon$ Epsilon  $\P$ $\rho$ Rho
$\Zeta$ $\zeta$ Zeta $\Sigma$ $\sigma$ Sigma
$\Eta$ $\eta$ Eta  $\T$ $\tau$ Tau
$\Theta$ $\theta$ Theta  $\Upsilon$ $\upsilon$ Upsilon
$\Iota$ $\iota$ Iota  $\Phi$ $\varphi$ Phi
$\Kappa$ $\kappa$ Kappa  $\chi$ Chi
$\Lambda$ $\lambda$ Lambda  $\Psi$ $\psi$ Psi
$\Mu$ $\mu$ Mu  $\Omega$ $\omega$ Omega
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CHAPTER 1  PRELIMINARIES  1
1.1  Sets and Functions  1
1.2  Mathematical Induction  12
1.3  Finite and Infinite Sets  16

CHAPTER 2  THE REAL NUMBERS  23
2.1  The Algebraic and Order Properties of $\mathbb{R}$  23
2.2  Absolute Value and the Real Line  32
2.3  The Completeness Property of $\mathbb{R}$  36
2.4  Applications of the Supremum Property  40
2.5  Intervals  46

CHAPTER 3  SEQUENCES AND SERIES  54
3.1  Sequences and Their Limits  55
3.2  Limit Theorems  63
3.3  Monotone Sequences  70
3.4  Subsequences and the Bolzano-Weierstrass Theorem  78
3.5  The Cauchy Criterion  85
3.6  Properly Divergent Sequences  91
3.7  Introduction to Infinite Series  94

CHAPTER 4  LIMITS  102
4.1  Limits of Functions  103
4.2  Limit Theorems  111
4.3  Some Extensions of the Limit Concept  116

CHAPTER 5  CONTINUOUS FUNCTIONS  124
5.1  Continuous Functions  125
5.2  Combinations of Continuous Functions  130
5.3  Continuous Functions on Intervals  134
5.4  Uniform Continuity  141
5.5  Continuity and Gauges  149
5.6  Monotone and Inverse Functions  153
CHAPTER 6  DIFFERENTIATION  161
       6.1  The Derivative  162
       6.2  The Mean Value Theorem  172
       6.3  L’Hospital’s Rules  180
       6.4  Taylor’s Theorem  188

CHAPTER 7  THE RIEMANN INTEGRAL  198
       7.1  Riemann Integral  199
       7.2  Riemann Integrable Functions  208
       7.3  The Fundamental Theorem  216
       7.4  The Darboux Integral  225
       7.5  Approximate Integration  233

CHAPTER 8  SEQUENCES OF FUNCTIONS  241
       8.1  Pointwise and Uniform Convergence  241
       8.2  Interchange of Limits  247
       8.3  The Exponential and Logarithmic Functions  253
       8.4  The Trigonometric Functions  260

CHAPTER 9  INFINITE SERIES  267
       9.1  Absolute Convergence  267
       9.2  Tests for Absolute Convergence  270
       9.3  Tests for Nonabsolute Convergence  277
       9.4  Series of Functions  281

CHAPTER 10  THE GENERALIZED RIEMANN INTEGRAL  288
       10.1  Definition and Main Properties  289
       10.2  Improper and Lebesgue Integrals  302
       10.3  Infinite Intervals  308
       10.4  Convergence Theorems  315

CHAPTER 11  A GLIMPSE INTO TOPOLOGY  326
       11.1  Open and Closed Sets in \( \mathbb{R} \)  326
       11.2  Compact Sets  333
       11.3  Continuous Functions  337
       11.4  Metric Spaces  341

APPENDIX A  LOGIC AND PROOFS  348

APPENDIX B  FINITE AND COUNTABLE SETS  357
APPENDIX C  THE RIEMANN AND LEBESGUE CRITERIA  360

APPENDIX D  APPROXIMATE INTEGRATION  364

APPENDIX E  TWO EXAMPLES  367

REFERENCES  370

PHOTO CREDITS  371

HINTS FOR SELECTED EXERCISES  372

INDEX  395